Journal of Sound and Vibration (1997) **202**(5), 739–747

A GENERAL FORMALISM FOR ANALYZING ACOUSTIC 2-PORT NETWORKS

R. Glav and M. Åbom

The Marcus Wallenberg Laboratory for Sound and Vibration Research, MWL, Department of Vehicle Engineering, Royal Institute of Technology, S-100 44 Stockholm, Sweden

(*Received* 6 *September* 1996)

1. INTRODUCTION

Low frequency (plane wave) sound propagation in flow duct systems is a problem of considerable practical interest. Over the years a number of papers have been published on the subject and nowadays several codes based on acoustic 2-port (or four-pole) methods are available, e.g., LAMPS [1] and SID [2]. For a background of the basic theory behind acoustic 2-ports, the papers by Davies [3, 4] and the book by Munjal [5] are recommended. In using 2-port methods a number of different formulations, depending on the choice of input and output state variables, are possible. The most commonly used formalism is obtained by choosing the acoustic pressure and volume velocity (or mass flow) at the two openings, respectively, as input and output state variables. This leads to the so-called transfer-matrix formalism, a method that is especially suited to treat problems with one preferred direction for the acoustic energy flow, corresponding to a number of cascade coupled 2-ports: e.g., as in intake and exhaust systems on automobiles [4, 5]. Although the transfer-matrix formalism is less suited for more general networks with arbitrary couplings, it has also been used for such applications [6]. For duct or pipe systems with a large number of branches and loops, alternative formalisms have been suggested by Frid [7] and Eversman [8]. Frid suggested a mobility-matrix formalism that uses the acoustic pressure at each of the two openings as input state variables. The output state variables are chosen as the corresponding acoustic volume velocities. With the mobility-matrix formalism it is possible to assemble easily the complete mobility-matrix for a complex network. However, this is not unique, since simple assembly procedures seem to exist independent of the choice of formalism, as demonstrated both by the paper by Eversman [8] and the results presented in this work. However, the mobility-matrix method will, with continuity in pressure and volume velocity assumed at each joint in a network, reduce the problem to the smallest possible number of unknowns (= number of joints). The method developed by Eversman [8] is based on a scattering-matrix formalism that uses travelling wave amplitudes as state variables. This type of formalism is attractive since it reflects the fundamental wave guide nature of the problem. Eversman's method is also well suited for generalization and can be said to be the basis for this work. One problem with all works cited above is the lack of general source models. Sources are normally only included as simple 1-port elements at a termination or sometimes coupled to a joint [7]. Very few published works seem to exist which allow for active 2-port elements in a network or for a general source arrangement at a joint. For a complete modelling of many flow duct systems the possibility to include sources inside the system is very important: e.g., to model in-duct fans and flow constrictions in pipes. It can be noted in this context that measurement methods to characterize both 1- and 2-port acoustic sources exist today. For a recent review of this subject together with a discussion of models for flow duct sources in general, see the paper by Bodén and Åbom [9].

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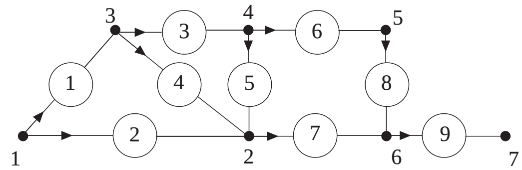

Figure 1. Example of a graph representation of a network of acoustical 2-ports. The network consists of nine 2-ports and seven joints (or node points). The arrows define the direction of mean flow for each 2-port. Note, for later reference that each 2-port is associated with two nodes.

It is the objective of this note to develop a general formalism for the analysis of sound propagation in 2-port networks. The formalism will allow for active 2-port elements as well as a general source arrangement at each joint: i.e., the joints will be regarded as active multi-ports.

2. ACOUSTICAL 2-PORT NETWORKS

2.1. *General*

A system of duct or pipes that connects a number of fluid machines and various muffler elements (which can be active or passive) can for low frequencies, when the acoustic field in the connecting ducts is of plane wave type, be represented as a network of acoustical 2-ports. These 2-ports are connected at joints (or nodes, in graph terminology) which are modelled as acoustic multi-ports with an order corresponding to the number of 2-ports that meet at a joint. The topological structure of any 2-port network and the corresponding physical system can be represented by a directional graph, as illustrated in Figure 1. With reference to Figure 1, it can be seen that, for instance, node no. 2 represents an acoustical 4-port (because four 2-ports meet at this node). Similarly, node number 3 is a 3-port, node number 7 is a 1-port and so forth. As illustrated in Figure 1, to describe a network each 2-port is assigned an integer number $1, 2, 3, \ldots, M$, where M is the total number of 2-ports. Similarly, the nodes (joints) are numbered from 1 to N, where N is the total number of nodes. To create a mathematical formalism for network analysis it is first necessary to write down the equations describing each 2-port and each node.

2.2. *Description of a* 2-*port*

The state of an acoustic 2-port can be completely defined by prescribing two pairs (one at each port) of plane wave state variables. If these state variables are taken as the travelling plane wave amplitudes as shown in Figure 2 then, in the frequency domain, the relationship defining an active 2-port can be written as [9]

$$\begin{pmatrix} p_{-1} \\ p_{-2} \end{pmatrix}_m = \mathbf{S}_m \begin{pmatrix} p_{+1} \\ p_{+2} \end{pmatrix}_m + \begin{pmatrix} p^s_{-1} \\ p^s_{-2} \end{pmatrix}_m, \tag{1}$$

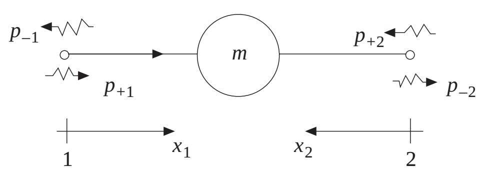

Figure 2. Definition of positive directions for 2-port number m, the port numbers have been chosen so that the direction of mean flow is from 1 to 2.

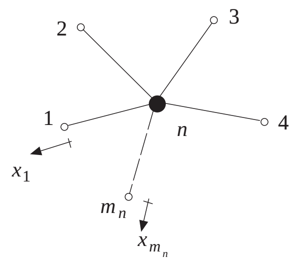

Figure 3. The definition of positive directions for node number n. The total number of 2-ports connected to this node is m_n, which corresponds to the order of the multi-port associated with the node. Note that the positive directions have been chosen away from the node in order to comply with the definition used for the 2-ports.

where $p_{+/-}$ is the complex (Fourier) pressure amplitude for a plane wave in the positive/negative direction, $\mathbf{S}$ is the $[2 \times 2]$ scattering-matrix, m denotes the number of the 2-port in a given network and the superscript s denotes source strength. The elements of the $\mathbf{S}$-matrix are given by

$$\mathbf{S}_m = \begin{bmatrix} S_{11,m} & S_{12,m} \\ S_{21,m} & S_{22,m} \end{bmatrix}. \tag{2}$$

For each of the M 2-ports of a network, equation (1) can be applied, which will give a total of $2M$ equations for $4M$ unknowns (all the amplitudes $p_{+/-}$). To find the extra $2M$ equations needed to obtain a complete problem, it is necessary to consider the equations describing the nodes.

2.3. *Description of a node*

In order to describe a node, the sign conventions illustrated in Figure 3 are adopted and it is assumed that each node behaves as an acoustic multi-port, with an order equal to the number of 2-ports connected to the node. This assumption implies that at each of the ports belonging to a node the sound field can be completely described in terms of plane waves. It is also assumed that each port of a node is associated with the same cross-section in the actual duct system as the 2-port port to which it is connected. Furthermore, these cross-sections should be chosen so that continuity of the state variables is satisfied. The requirements for plane waves and continuity will restrict the ways in which the cross-sections defining the ports can be chosen in a duct system. The following relationship can now be written down to describe the behaviour of a node in the frequency domain:

$$\mathbf{S}_+^n \mathbf{p}_+^n = \mathbf{S}_-^n \mathbf{p}_-^n + \mathbf{p}^{ns}. \tag{3)*$$

Here $\mathbf{p}_{+/-}$ are $[m_n \times 1]$ column vectors, where the kth element is the complex amplitude for a plane wave in the positive/negative direction at port number k and $\mathbf{S}_{+/-}$ are $[m_n \times m_n]$ scattering matrices associated with the positive/negative waves. The introduction of a scattering matrix for each direction of propagation in equation (3) is somewhat arbitrary, but is convenient because it means that matrix inversions can be avoided when deriving the multi-port equations for a node (see the Appendix). Of course, if preferred it is certainly

* Note that in equation (3) the source strength vector (with superscript s) cannot be interpreted as a wave in the positive direction and accordingly the subscript + has been omitted.

possible directly to extend the definition in equation (2) to the multi-port case. For later use, the following notation for the elements of the $\mathbf{S}_{+/-}$ matrices is also introduced: S^n_{-kl} and S^n_{+kl}. For each of the N nodes of a network equation (3) can be applied, and this will give a total of

$$\sum_{n=1}^{N} m_n = 2M$$

equations, since each 2-port belongs to two nodes; see Figure 1. In other words, if equations (1) and (3) are applied for all 2-ports and nodes, respectively, in a network, a total of $4M$ equations for the $4M$ unknown pressure amplitudes are obtained.

2.4. *Derivation of the complete network equation*

In this section, the equations for the 2-ports and the nodes will be assembled into a complete network equation. First, all the unknown pressure amplitudes of the network are put into two $[2M \times 1]$ column vectors,

$$\mathbf{p}^c_- = \begin{bmatrix} \begin{pmatrix} p_{-1} \\ p_{-2} \end{pmatrix}_1 \\ \vdots \\ \begin{pmatrix} p_{-1} \\ p_{-2} \end{pmatrix}_M \end{bmatrix} \quad \text{and} \quad \mathbf{p}^c_+ = \begin{bmatrix} \begin{pmatrix} p_{+1} \\ p_{+2} \end{pmatrix}_1 \\ \vdots \\ \begin{pmatrix} p_{+1} \\ p_{+2} \end{pmatrix}_M \end{bmatrix}. \tag{4, 5}$$

By using these vectors all the 2-port equations for a network can formally be written as one matrix equation,

$$\mathbf{p}^c_- = \mathbf{S}^c\mathbf{p}^c_+ + \mathbf{p}^{cs}_-, \tag{6}$$

where $\mathbf{p}^{cs}_-$ is a column vector containing the source strengths and

$$\mathbf{S}^c = \begin{bmatrix} \mathbf{S}_1 & 0 & \cdot & \cdot & 0 \\ 0 & \mathbf{S}_2 & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & \mathbf{S}_M \end{bmatrix}.$$

The next step is to rewrite the node equations by introducing the state vectors for the complete network (equations (4) and (5)). To do this it is necessary to have the relationship between the state vectors used at a certain node and the state vectors for the complete network. This relationship can be defined by introducing a $[m_n \times 2M]$ projection matrix $\mathbf{G}^n$, which satisfies

$$\mathbf{p}^n_{+/-} = \mathbf{G}^n\mathbf{p}^c_{+/-}. \tag{7}$$

Although not necessary, but as it will facilitate the final matrix assembling in the problem, it will now be assumed that the local ordering of the ports at each node have been chosen so that it corresponds to (in growing order) the numbers of the 2-ports connected to the node. As an example, when using this convention, the state vectors associated with node number 2 in the network illustrated in Figure 1, are

$$\mathbf{p}_-^2 = \begin{bmatrix} p_{-2,2} \\ p_{-2,4} \\ p_{-2,5} \\ p_{-1,7} \end{bmatrix} \qquad \text{and} \qquad \mathbf{p}_+^2 = \begin{bmatrix} p_{+2,2} \\ p_{+2,4} \\ p_{+2,5} \\ p_{+1,7} \end{bmatrix},$$

where the first index defines which 2-port port (1 or 2) that is connected and the second index denotes the number of the connected 2-port. From this result it follows directly that the [4 × 18] projection matrix for node number 2 is given by

$$\mathbf{G}^2 = \begin{bmatrix} 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \end{bmatrix}.$$

By introducing equation (7) in equation (3), the following result is obtained for node number n:

$$\mathbf{S}_+^n \mathbf{G}^n \mathbf{p}_+^c = \mathbf{S}_-^n \mathbf{G}^n \mathbf{p}_-^c + \mathbf{p}^{ns}. \tag{8}$$

By using equation (6) it is possible to rewrite equation (8) so that it involves only the amplitudes in the positive directions as unknowns:

$$\mathbf{S}_+^n \mathbf{G}^n \mathbf{p}_+^c = \mathbf{S}_-^n \mathbf{G}^n \mathbf{S}^c \mathbf{p}_+^c + \mathbf{S}_-^n \mathbf{G}^n \mathbf{p}_-^{cs} + \mathbf{p}^{ns}. \tag{9}$$

If equation (9) is applied at all nodes a total of $2M$ equations (as noted above) are obtained, which corresponds to the number of unknowns in the vector $\mathbf{p}_+^c$. By assembling these equations a complete matrix equation for the network can be generated. Formally, this assembly procedure can be described by

$$\left[\sum_n (\mathbf{S}_+^n \mathbf{G}^n - \mathbf{S}_-^n \mathbf{G}^n \mathbf{S}^c)\right]\mathbf{p}_+^c = \left[\sum_n \mathbf{S}_-^n \mathbf{G}^n\right]\mathbf{p}_-^{cs} + \left(\sum_n \mathbf{p}^{ns}\right), \tag{10}$$

where the summation symbolizes an assembly procedure which in the nth step adds the next m_n rows to the matrix equation. This means that for $n = 1$ the rows 1 to m_1 are added, for $n = 2$ the rows $m_1 + 1$ to $m_1 + m_2$ are added and so on until the complete matrix with $2M$ rows has been assembled. Due to the special structure of the projection matrices, it is possible to express the assembly procedure in terms of a simple algorithm which will require no matrix multiplications. This question will be addressed in the next section.

2.5. *An algorithm for assembling the network matrix equation*

In order to derive this algorithm it is necessary to introduce a formalism to describe the topology of a 2-port network. This can be done by defining the so-called graph matrix

$\mathcal{G}$. The $\mathcal{G}$-matrix has a row for each node of a network and a column for each 2-port. The order of the matrix is therefore $[N \times M]$ and the elements are defined by

$$g(n, m) = \left\{ \begin{array}{rl} 1, & \text{if port no. 1 of the } m\text{th 2-port is coupled to the } n\text{th node} \\ 0, & \text{if the } m\text{th 2-port is } \textit{not} \text{ coupled to the } n\text{th node} \\ -1, & \text{if port no. 2 of the } m\text{th 2-port is coupled to the } n\text{th node} \end{array} \right\}. \tag{11}$$

The relationship between the $\mathcal{G}$-matrix and the earlier discussed projection matrices matrices $\mathcal{G}^n$ is easily seen. As an example, the $\mathcal{G}$-matrix for the network illustrated in Figure 1 is given by

$$\mathcal{G} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

By studying the matrix multiplications in equation (10) it is straightforward to derive the following simple rules for assembling the different parts of the network matrix equation. By inspection of the nth row of the $\mathcal{G}$-matrix the next m_n rows of the matrix equation are generated, where m_n is the number of elements in the nth row different from 0. The elements of the kth row of these m_n new rows are, for the left side of equation (10),

$$\underset{\text{step } n:}{\left[\sum_n (\mathbf{S}_+^n \mathbf{G}^n - \mathbf{S}_-^n \mathbf{G}^n \mathbf{S}^c)\right]} m_n \overrightarrow{\text{new}} \text{ rows}$$

$$\overset{\text{column no.:}}{\left\{ \begin{array}{ccc} (2m-1) & 2m & \\ S^n_{+kl} - S^n_{-kl} S_{11,m} & -S^n_{-kl} S_{12,m}, & g(n, m) > 0 \\ -S^n_{-kl} S_{21,m} & S^n_{+kl} - S^n_{-kl} S_{22,m}, & g(n, m) < 0 \end{array} \right\}}, \tag{12}$$

and, for the right side,

$$\underset{\text{step } n:}{\left[\sum_n \mathbf{S}_-^n \mathbf{G}^n\right]} \; m_n \overrightarrow{\text{new}} \text{ rows} \overset{\text{column no.:}}{\left\{ \begin{array}{ccc} (2m-1) & 2m & \\ S^n_{-kl} & 0, & g(n, m) > 0 \\ 0 & S^n_{-kl}, & g(n, m) < 0 \end{array} \right\}}, \tag{13}$$

where l denotes that $g(n, m)$ is the lth element different from 0 in row n and $k, l = 1, 2, 3, \ldots, m_n$. The remaining columns for the rows generated by node number n will only contain zeros. Since the node source vector on the right side of equation (10) is in a form which allows direct assembling, it has been left out in the description. Equation (10) plus the assembling procedure described in this section represents the solution to the problem of obtaining a general equation for analysis of acoustic 2-port networks. When the resulting network equation is solved, the state vector $\mathbf{p}_+^c$ for the complete network is obtained. From this and the 2-port equation for the complete network (equation (6)) the state vector $\mathbf{p}^c_-$ can be calculated. When both $\mathbf{p}^c_+$ and $\mathbf{p}^c_-$ have been calculated the complete acoustic state of the network is known and all quantities of interest can be obtained: e.g., acoustic energy flows in various parts.

3. DISCUSSION

The formalism presented here is more general than earlier treatments of acoustical 2-port networks. In particular, the formalism allows for arbitrary source arrangements since both the 2-ports as well as the joints (the nodes) in a network are allowed to be active. Regarding the size of the resulting matrix equation, the suggested formalism is more efficient than the method suggested by Eversman [8]. In particular for a completely general network the formalism described here reduces the problem to $2M$ unknowns, while Eversman's method leads to $4M$ unknowns (both $\mathbf{p}^c_+$ and $\mathbf{p}^c_-$). For a general network the assembled matrix on the left side of equation (10) will be full, which implies that the number of operations necessary to solve the problem will be proportional to M^3. Therefore compared to Eversman's method the number of operations necessary to solve a general network will be reduced (for large systems) by a factor of one eighth. This conclusion is valid even if it is taken into account that (i) the assembling procedure of the method described here is slightly more complex than in Eversman's case, and (ii) to obtain the complete acoustic state the vector $\mathbf{p}^c_-$ must also be calculated. The first point* will only add a number of operations proportional to M, and the second to M^2; which for a large system will be negligible compared to M^3. For simpler systems which consists only of connected straight pipes, modelled as plane wave acoustic elements, it can be shown that Eversman's method gives the same number of unknowns as the method suggested here. Regarding the mobility-matrix method suggested by Frid [7], it gives a number of unknowns equal to N which, since $M \geqslant N - 1$ for all 2-port networks, always is less than $2M$. This is the minimum number of unknowns possible. However, the method is based on the assumption of continuity of pressure and volume velocity at the nodes, and both the method suggested here and Eversman's method are works without this limitation.

Regarding how to find a suitable model for a node or a 2-port in a given network references [2–9] should be consulted. One simple model for nodes is also discussed in the Appendix.

ACKNOWLEDGMENT

This work was carried out under contract from the National Swedish Board for Technical Development (NUTEK); project no. P1850-3.

REFERENCES

1. K. S. Peat 1995 *Proceedings of Euro-Noise* '95, 791–796. LAMPS software for the acoustic analysis of silencers.
2. R. Glav 1994 *Doctoral thesis from MWL, Department of Vehicle Engineering, Royal Institute of Technology, Stockholm*. On acoustic modelling of silencers.
3. P. O. A. L. Davies 1988 *Journal of Sound and Vibration* **124**, 91–115. Practical flow duct acoustics.
4. P. O. A. L. Davies 1996 *Journal of Sound and Vibration* **190**, 677–712. Piston engine intake and exhaust system design.
5. M. L. Munjal 1987 *Acoustics of Ducts and Mufflers*. New York: Wiley-Interscience.

* It is assumed that there is an upper (fixed) limit for the multi-port order of the nodes, i.e., max (m_n) exists.

6. C. W. S. To 1984 *Journal of Sound and Vibration* **96**, 175–194. The acoustic simulation and analysis of complicated reciprocating compressor piping systems, I: Analysis technique and parameter matrices of acoustic elements.
7. A. Frid 1989 *Journal of Sound and Vibration* **133**, 423–438. Fluid vibration in piping systems-a structural mechanics approach, I: theory.
8. W. Eversman 1987 *Transactions of the American Society of Mechanical Engineers Journal of Vibrations, Acoustics, Stress and Reliability in Design* **109**, 168–177. A systematic procedure for the analysis of multiply branched acoustic transmission lines.
9. H. Bodén and M. Åbom 1995 *Acta Acustica* **3**, 549–560. Modelling of fluid machines as sources of sound in duct and pipe systems.

APPENDIX: A SIMPLE MODEL FOR A NODE

Here a multi-port model for a node (see Figure 3) will be derived. To simplify the problem the following idealizations are made: (i) only plane waves are considered; (ii) the acoustic pressure is continuous across the node; (iii) the total acoustic mass flow is conserved at the node; (iv) the node is connected to a 1-port source. The first assumption means that inertia effects associated with non-propagating higher order modes are neglected. The second and third assumptions imply that the node is acoustically compact and that compressibility effects and entropy fluctuations are negligible. The fourth assumption implies that the pressure p^n at the node and the total mass flow q^n out from the node, can be related by the 1-port equation [9]

$$p^n = p_0^{ns} - Z_s^n q^n, \tag{A1}$$

where p_0^{ns} is the source strength and Z_s^n is the source impedance associated with node number n. Based on these assumptions, the following equations can be written down

$$\left\{\begin{array}{c} \dfrac{1}{Z_1^n}(p_{1+}^n - p_{1-}^n) + \cdots + \dfrac{1}{Z_{m_n}^n}(p_{m_n+}^n - p_{m_n-}^n) = q^n \\ p_{1+}^n + p_{1-}^n = p_{2+}^n + p_{2-}^n \\ \vdots \\ p_{(m_n-1)+}^n + p_{(m_n-1)-}^n = p_{m_n+}^n + p_{m_n-}^n \end{array}\right\}. \tag{A2}$$

Here Z_k^n is the characteristic impedance* at port number k. The continuity of pressure across the node also implies that

$$p_{1+}^n + p_{1-}^n = p^n. \tag{A3}$$

By using equations (A1) and (A3) it is now possible to rewrite equation (A2) in a form corresponding to the multi-port definition given in equation (3):

$$\begin{bmatrix} \dfrac{1}{Z_1^n} + \dfrac{1}{Z_s^n} & \dfrac{1}{Z_2^n} & \cdot & \cdot & \dfrac{1}{Z_{m_n}^n} \\ 1 & -1 & 0 & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} p_{1+}^n \\ \cdot \\ \cdot \\ \cdot \\ p_{m_n+}^n \end{bmatrix} = \begin{bmatrix} \dfrac{1}{Z_1^n} - \dfrac{1}{Z_s^n} & \dfrac{1}{Z_2^n} & \cdot & \cdot & \dfrac{1}{Z_{m_n}^n} \\ -1 & 1 & 0 & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} p_{1-}^n \\ \cdot \\ \cdot \\ \cdot \\ p_{m_n-}^n \end{bmatrix} + p_0^{ns} \begin{bmatrix} \dfrac{1}{Z_s^n} \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}. \tag{A4}$$

* When acoustic pressure and mass flow are used as state variables the characteristic impedance (= the impedance experienced by a propagating plane wave) is given by c/A, where c is the speed of sound and A is the cross-sectional area of the duct.

The advantage with the definition given in equation (3) is clear from this result: i.e., because a matrix inversion (of $\mathbf{S}_+^n$) and then a multiplication (with a full matrix) would be necessary if the standard multi-port definition [9] had been used. It can also be noted that for this simple model of a node the assembling of the network matrix equation can be simplified since the matrices $\mathbf{S}_+^n$ and $\mathbf{S}_-^n$ have almost a band structure. This means that in equations (12) and (13) for $k \geqslant 2$, only elements with $l = k - 1$ and $l = k$ need to be considered.